Division of Strength of Materials and Structures
Faculty of Power and Aeronautical Engineering

Finite element method (FEM1)

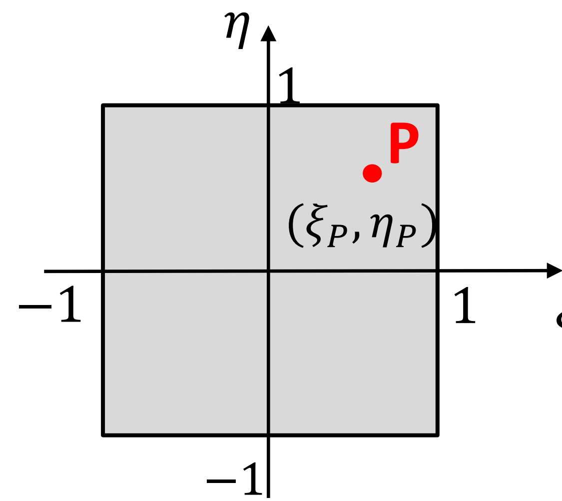
Lecture 4B. QUAD-8node element - example

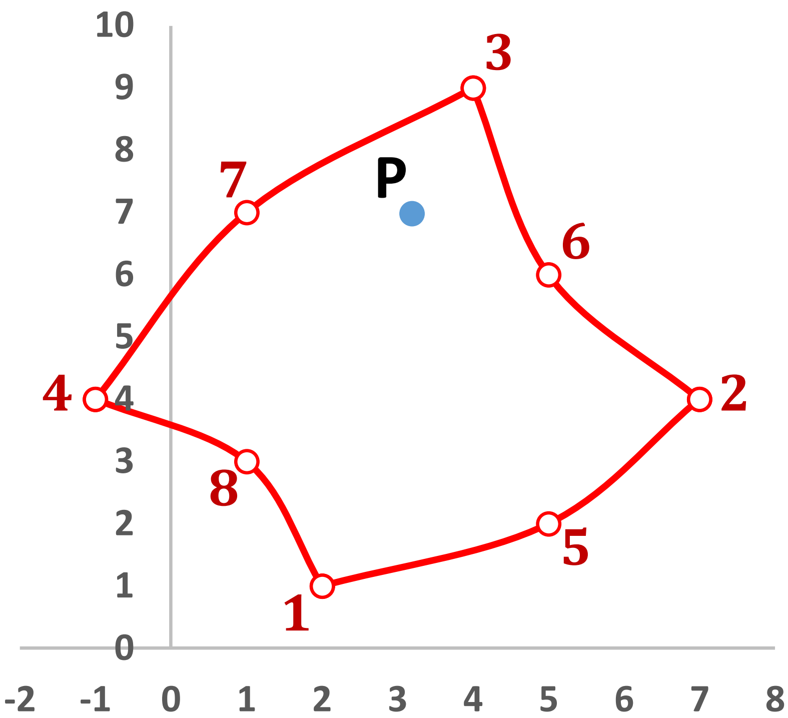
03.2025

Example 1. QUAD-8node: Find coordinates and det [J] at point P.

coordinates in the natural system:

$$\xi_P = \eta_P = \frac{1}{\sqrt{3}}$$

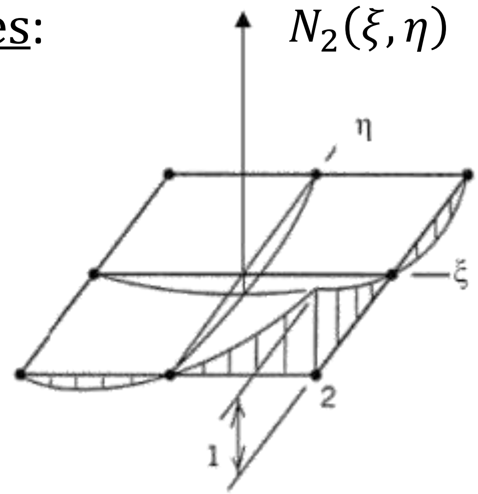




$$\underset{8\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 7 \\ 4 \\ -1 \\ 5 \\ 5 \\ 1 \\ 1 \end{Bmatrix} \quad ; \quad \underset{8\times 1}{\{y_i\}_e} = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \\ 9 \\ 4 \\ 2 \\ 6 \\ 7 \\ 3 \end{Bmatrix}$$

Shape functions of an 8-node quadrilateral element

corner nodes:



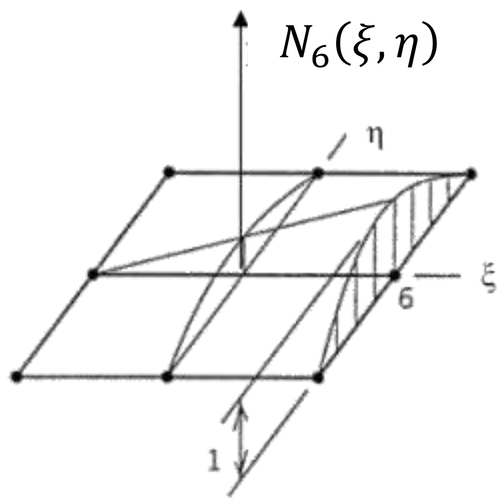
$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

midside nodes:



$$N_5(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_6(\xi,\eta) = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_7(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_8(\xi,\eta) = \frac{1}{2}(1-\xi)(1-\eta^2)$$

Example shape function and its derivative for node 1

$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta)\,\frac{\partial\left[(1-\xi)\,(1+\xi+\eta)\right]}{\partial \xi} =$$

$$= -\frac{1}{4}(1-\eta)\left[\frac{\partial(1-\xi)}{\partial \xi}\,(1+\xi+\eta) + \frac{\partial(1+\xi+\eta)}{\partial \xi}\cdot(1-\xi)\right] =$$

$$= -\frac{1}{4}(1-\eta)\left[-1\cdot(1+\xi+\eta) + 1\cdot(1-\xi)\right] =$$

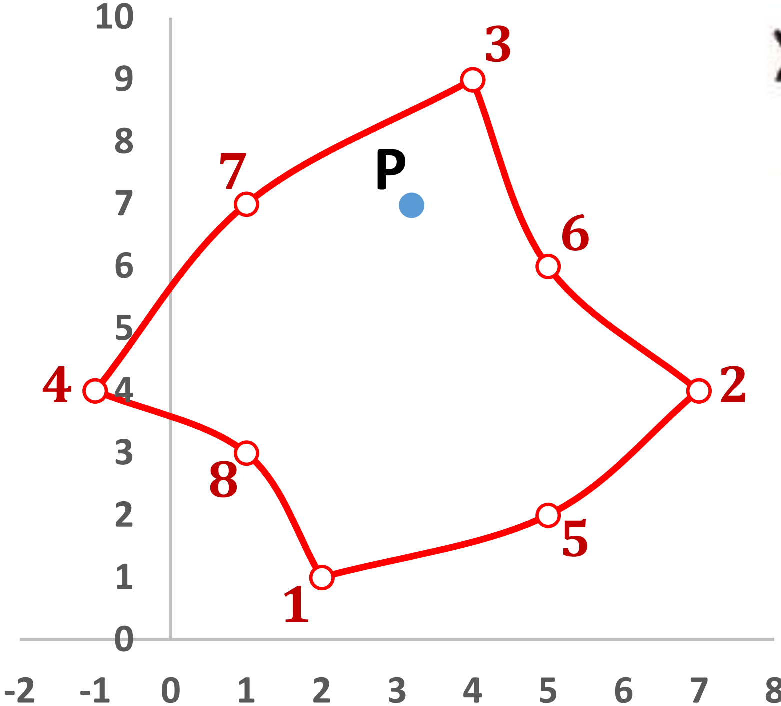
$$= -\frac{1}{4}(1-\eta)\left[-1-\xi-\eta+1-\xi\right] = -\frac{1}{4}(1-\eta)\left[-2\xi-\eta\right] =$$

$$= \frac{1}{4}(1-\eta)(2\xi+\eta)$$

Shape functions and their derivatives for an 8-node element

i	$\frac{\partial N_i}{\partial \xi}$	$\frac{\partial N_i}{\partial \eta}$
1	$\frac{1}{4}(1-\eta)(2\xi+\eta)$	$\frac{1}{4}(1-\xi)(\xi+2\eta)$
2	$\frac{1}{4}(1-\eta)(2\xi-\eta)$	$\frac{1}{4}(1+\xi)(2\eta-\xi)$
3	$\frac{1}{4}(1+\eta)(2\xi+\eta)$	$\frac{1}{4}(1+\xi)(\xi+2\eta)$
4	$\frac{1}{4}(1+\eta)(2\xi-\eta)$	$\frac{1}{4}(1-\xi)(2\eta-\xi)$
5	$-(1-\eta)\xi$	$-\frac{1}{2}(1-\xi^2)$
6	$\frac{1}{2}(1-\eta^2)$	$-(1+\xi)\cdot\eta$
7	$-(1+\eta)\xi$	$\frac{1}{2}(1-\xi^2)$
8	$-\frac{1}{2}(1-\eta^2)$	$-(1-\xi)\cdot\eta$

Coordinates of point P approximated by shape functions



Cartesian coordinates:

$$x_p = \underset{1\times 8}{\lfloor N(\xi_p, \eta_p) \rfloor} \cdot \underset{8\times 1}{\{x_i\}_e} = 3.1925 \text{ mm}$$

$$y_p = \underset{1\times 8}{\lfloor N(\xi_p, \eta_p) \rfloor} \cdot \underset{8\times 1}{\{y_i\}_e} = 6.9761 \text{ mm}$$

$$\underset{8\times 1}{\{x_i\}_e} = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 7 \\ 4 \\ -1 \\ 5 \\ 5 \\ 1 \\ 1 \end{Bmatrix} \quad ; \quad \{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \\ 9 \\ 4 \\ 2 \\ 6 \\ 7 \\ 3 \end{Bmatrix}$$

Determinant of the Jacobian matrix for point P

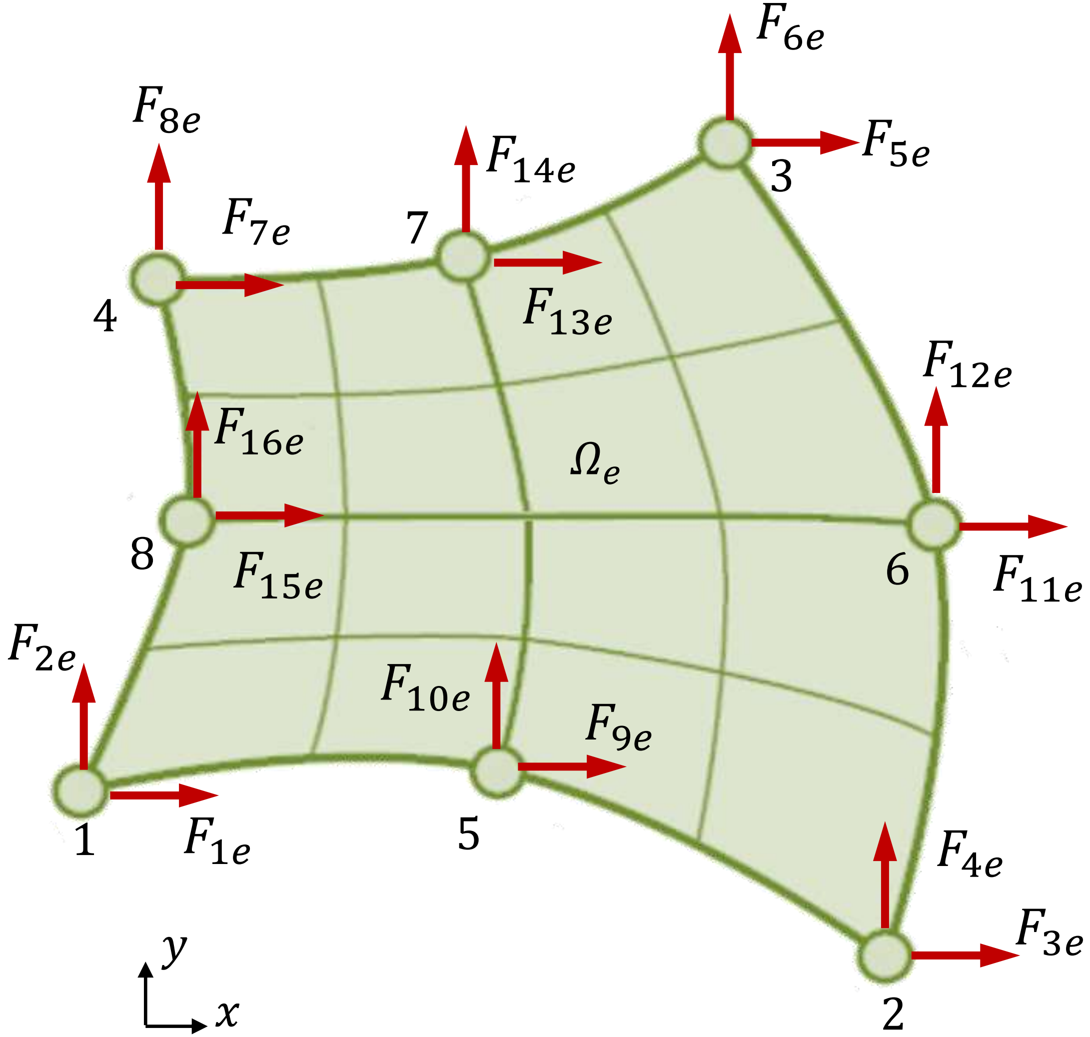
$$det[J] = \left(\underset{1\times 8}{\frac{\partial[N(\xi,\eta)]}{\partial\xi}}\ \underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial[N(\xi,\eta)]}{\partial\eta}}\ \underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\frac{\partial[N(\xi,\eta)]}{\partial\xi}}\ \underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial[N(\xi,\eta)]}{\partial\eta}}\ \underset{8\times 1}{\{x_i\}_e}\right)$$

$$\{x_i\}_e = \begin{Bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_8 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 7 \\ 4 \\ -1 \\ 5 \\ 5 \\ 1 \\ 1 \end{Bmatrix} \quad ; \quad \{y_i\}_e = \begin{Bmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_8 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 4 \\ 9 \\ 4 \\ 2 \\ 6 \\ 7 \\ 3 \end{Bmatrix}$$

ξ	0.57735							
η	0.57735							
i	1	2	3	4	5	6	7	8
N	-0.09623	-0.16667	0.096225	-0.16667	0.140883	0.525783	0.525783	0.140883
dN/dξ	0.183013	0.061004	0.683013	0.227671	-0.24402	0.333333	-0.91068	-0.33333
dN/dη	0.183013	0.227671	0.683013	0.061004	-0.33333	-0.91068	0.333333	-0.24402
det[J]	**9.821367**							

Equivalent load vector in the 8-node quadrilateral element

$$\underset{16\times 1}{\lfloor F \rfloor_e}$$

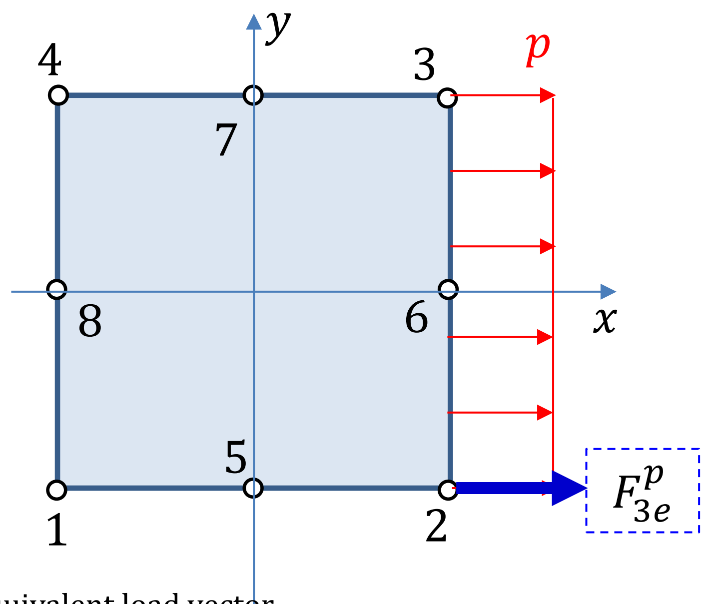


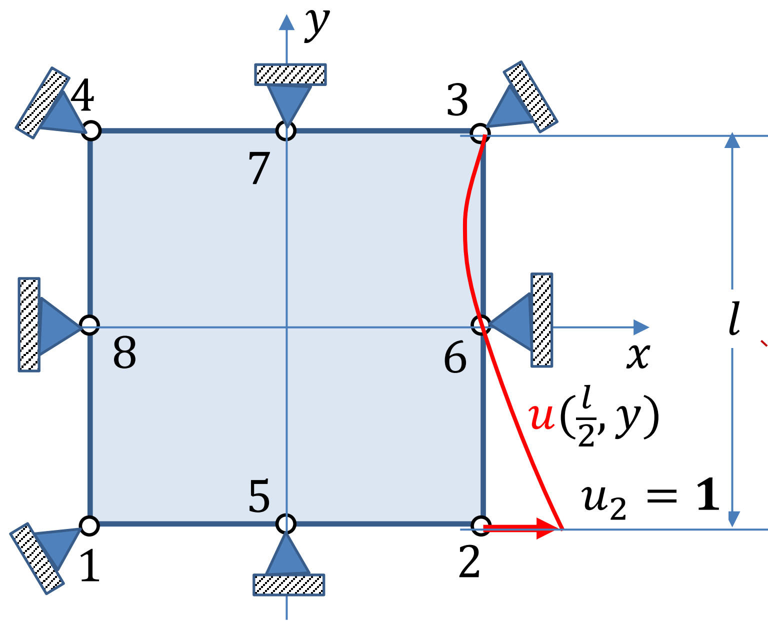
$$\{F\}_e = \begin{Bmatrix} F_1 \\ F_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ F_{16} \end{Bmatrix}_e$$

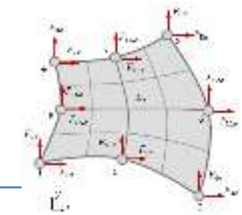
➔ Determination of the equivalent force in an 8-node element

Example 2a Determination of the equivalent force in an 8-node element due to surface load

We will look for the equivalent force at node 2 resulting from the constant load p acting on the edge2-3







$$dy = \frac{l}{2}\, d\eta$$

equivalent load vector *due to* surface load :

$$\lfloor F^p \rfloor_e = t_e \int_0^l \lfloor p \rfloor [N]\, ds$$

The work of the equivalent force F_{3e}^p on displacement 1

The work of load $p(x,y)$ on displacement $u(x,y)$

$$F_{3e}^p \cdot 1 = t_e \int_0^l p(x,y)\, u(\tfrac{l}{2}, y)\, dy$$

$$F_{3e}^p = t_e \int_0^l p\, N_2\, dy$$

$$N_2(\xi,\eta) = -\tfrac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

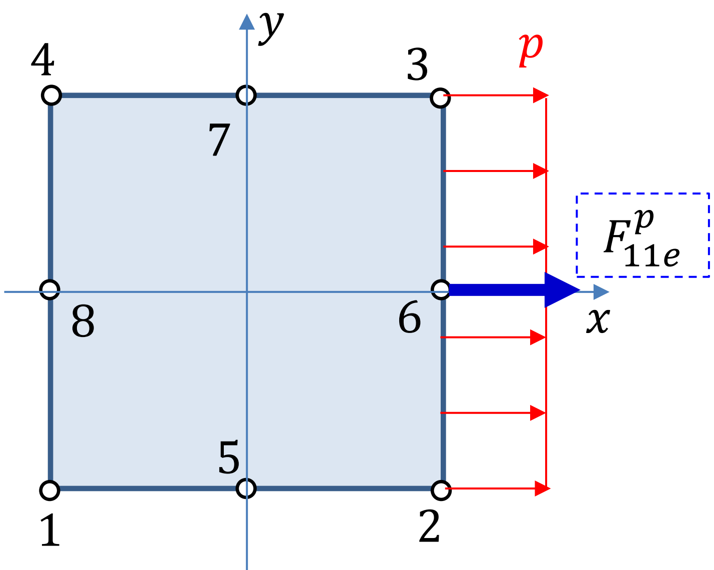
$$N_2(1,\eta) = -\tfrac{1}{2}(1-\eta)\eta$$

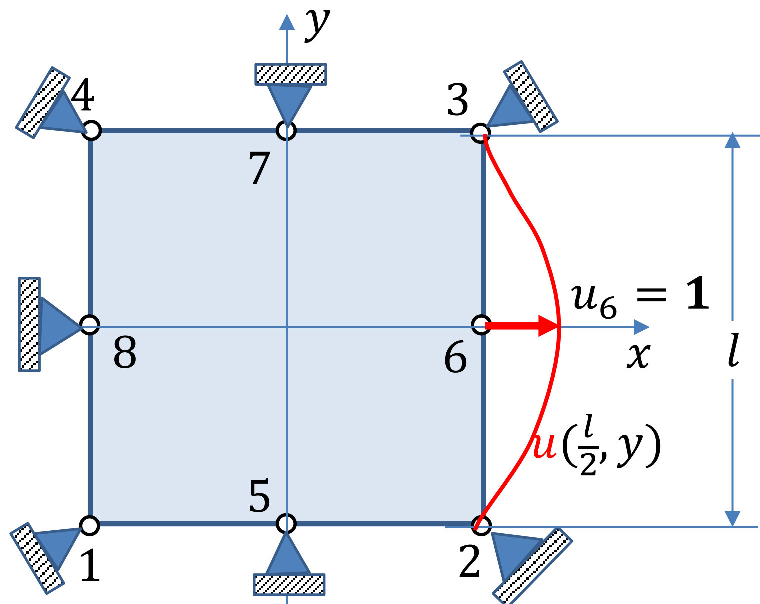
$$F_{3e}^p = t_e p \int_{-1}^{1} \tfrac{1}{2}(\eta - 1)\eta\; \tfrac{l}{2}\, d\eta =$$

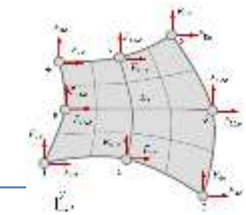
$$= \frac{p\,l}{4} t_e \left(\frac{1}{3}\eta^3 - \frac{1}{2}\eta^2\right)\Big|_{-1}^{1} = \frac{p\,l}{6} t_e$$

Example 2b Determination of the equivalent force in an 8-node element due to surface load

We will look for the equivalent force at node **6** resulting from the constant load p acting on the edge 2-3







$dy = \frac{l}{2}\, d\eta$

equivalent load vector *due to surface load* :

$$\lfloor F^p \rfloor_e = t_e \int_0^l \lfloor p \rfloor [N]\, ds$$

The work of the equivalent force F^p_{11e} on displacement 1

The work of load $p(x,y)$ on displacement $u(x,y)$

$$F^p_{11e} \cdot 1 = t_e \int_0^l p(x,y)\, u(\tfrac{l}{2}, y)\, dy$$

$$F^p_{11e} = t_e \int_0^l p\, N_6\, dy$$

$$N_6(\xi,\eta) = \tfrac{1}{2}(1+\xi)(1-\eta^2)$$

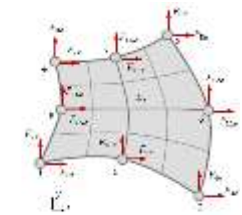
$$N_6(1,\eta) = 1 - \eta^2$$

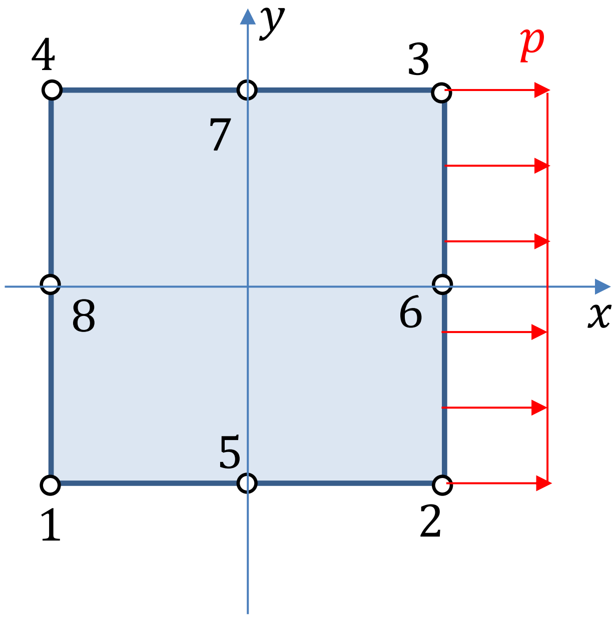
$$F^p_{11e} = t_e p \int_{-1}^{1} (1-\eta^2)\, \tfrac{l}{2}\, d\eta =$$

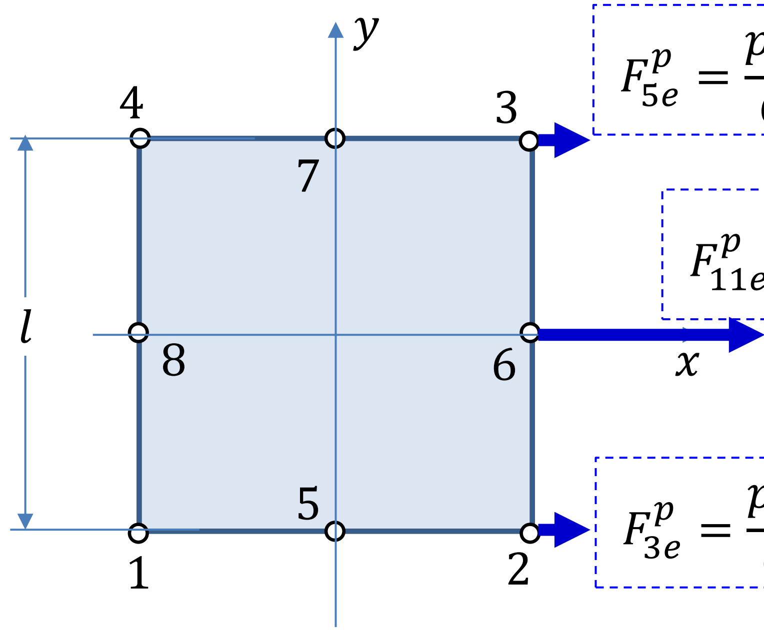
$$= \frac{p\, l}{2} t_e \left(\eta - \frac{1}{3}\eta^3\right)\Big|_{-1}^{1} = \frac{2p\, l}{3} t_e$$

Example 2 Determination of the equivalent force in an 8-node element due to surface load

Equivalent forces resulting from the constant load p acting on the edge 2-3







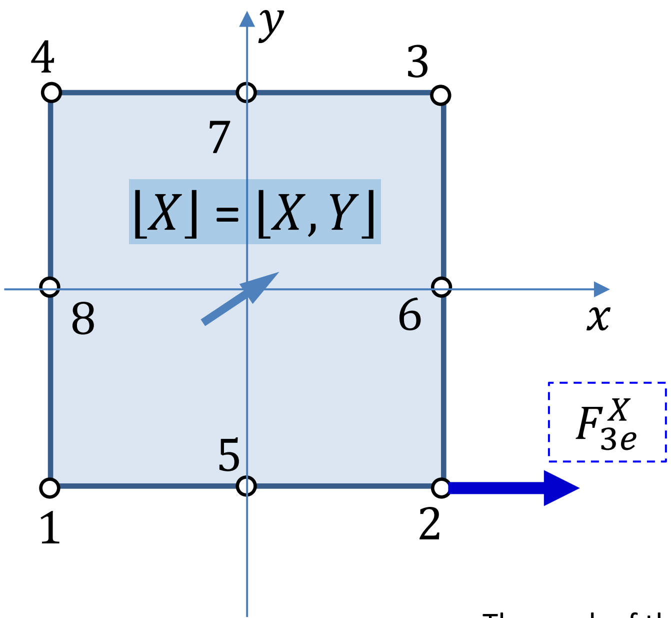
$$F_{5e}^{p} = \frac{p\,l}{6} t_e$$

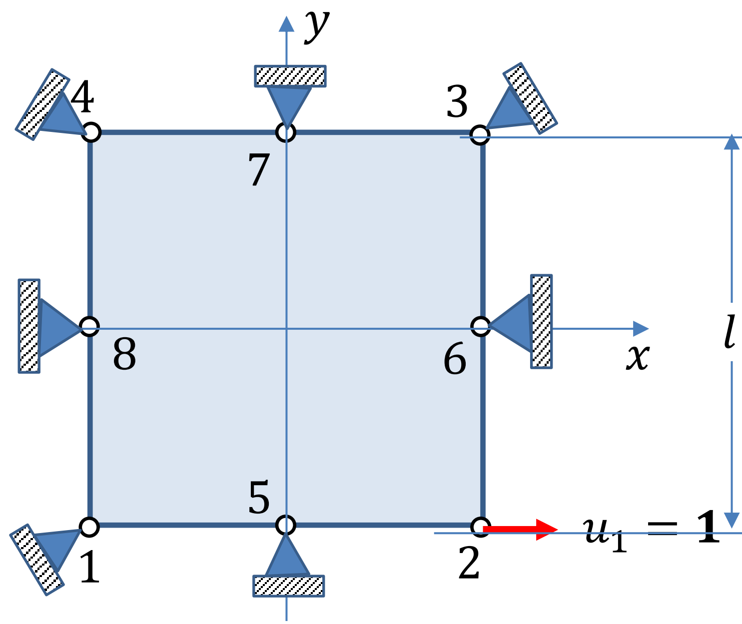
$$F_{11e}^{p} = \frac{4p\,l}{6} t_e$$

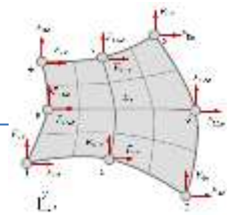
$$F_{3e}^{p} = \frac{p\,l}{6} t_e$$

Example 3 Determination of the equivalent force in an 8-node element due to mass loads

We will look for the equivalent force at node **2** resulting from the constant mass load ***X***







$$dy = \frac{l}{2}\,d\eta$$

$$dx = \frac{l}{2}\,d\xi$$

equivalent load vector *due to* mass load:

$$\lfloor F^X \rfloor_e = t_e \int_{A_e} \lfloor X \rfloor [N] dA_e$$

The work of the equivalent force F_{3e}^X acting on displacement 1

The work of the mass load X acting on displacement $u(x, y)$

$$F_{3e}^X \cdot 1 = t_e \int_{A_e} X\, u(x,y)\, dA$$

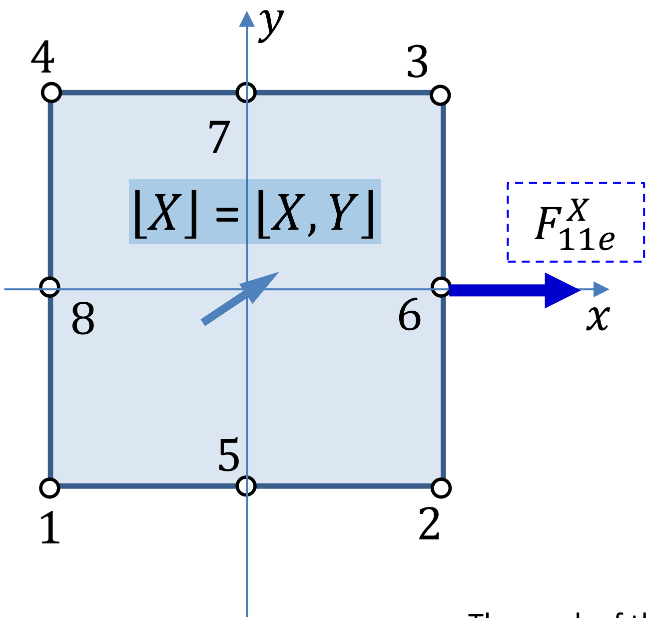
$$F_{3e}^X = t_e \int_{A_e} X\, N_2\, dA$$

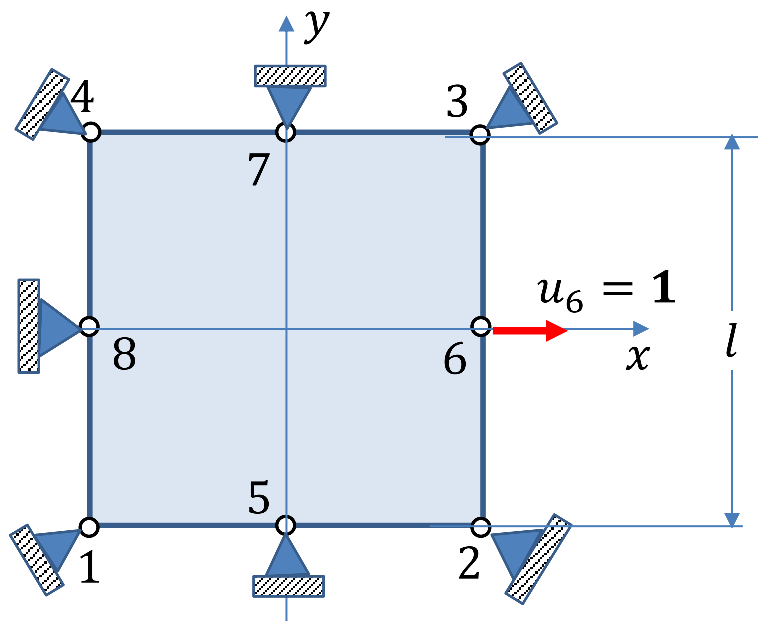
$$N_2(\xi,\eta) = -\tfrac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$F_{3e}^X = t_e X \int_{-1}^{1}\int_{-1}^{1} -\tfrac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)\,\tfrac{l}{2}\,d\xi\,\tfrac{l}{2}\,d\eta = -\,\tfrac{1}{12}\,t_e X l^2$$

Example 3 Determination of the equivalent force in an 8-node element due to mass loads

We will look for the equivalent force at node **6** resulting from the constant mass load $\boldsymbol{X}$





$$dy = \frac{l}{2}\, d\eta$$

$$dx = \frac{l}{2}\, d\xi$$

equivalent load vector *due to* mass load: :

$$\lfloor F^X \rfloor_e = t_e \int_{A_e} \lfloor X \rfloor [N] dA_e$$

$$N_6(\xi,\eta) = \tfrac{1}{2}(1+\xi)(1-\eta^2)$$

The work of the equivalent force F^X_{11e} acting on displacement 1

The work of the mass load X acting on displacement $u(x,y)$

$$F^X_{11e} \cdot 1 = t_e \int_{A_e} X\, u(x,y)\, dA$$

$$F^X_{11e} = t_e \int_{A_e} X\, N_6\, dA$$

$$F^X_{11e} = t_e X \int_{-1}^{1} \int_{-1}^{1} \tfrac{1}{2}(1+\xi)(1-\eta^2)\, \tfrac{l}{2}\, d\xi\, \tfrac{l}{2}\, d\eta = \tfrac{1}{3} t_e X l^2$$

Example 3 Determination of the equivalent forces in an 8-node element due to mass loads

Equivalent forces resulting from the constant mass load $\boldsymbol{X} = \lfloor X, Y \rfloor$